The Cauchy-Riemann Equation on Pseudoconcave Domains with Applications

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Joint work with Siqi Fu and Christine Laurent-Thiébaut

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Outline

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The $\overline{\partial}$-problem

Let Ω be a domain in $\mathbb{C}^n$ (or a complex manifold), $n \geq 2$. Given a (p, q)-form g such that $\overline{\partial} g = 0$, find a $(p, q-1)$-form u such that $\overline{\partial} u = g$.

If g is in $C^\infty_{p,q}(\Omega)$ (or $g \in C^\infty_{p,q}(\overline{\Omega})$), one seeks $u \in C^\infty_{p,q-1}(\Omega)$ (or $u \in C^\infty_{p,q-1}(\overline{\Omega})$).

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$$H^{p,q}(\Omega) = \frac{\ker\{\overline{\partial} : C^\infty_{p,q}(\Omega) \to C^\infty_{p,q+1}(\Omega)\}}{\text{range}\{\overline{\partial} : C^\infty_{p,q-1}(\Omega) \to C^\infty_{p,q}(\Omega)\}} \quad (H^{p,q}(\overline{\Omega}))$$

- Obstruction to solving the $\overline{\partial}$-problem on Ω.
- Natural topology arising as quotients of Fréchet topologies on $\ker(\overline{\partial})$ and range$(\overline{\partial})$.
- This topology is Hausdorff iff range$(\overline{\partial})$ is closed in $C^\infty_{p,q}(\Omega)$

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Two ways to close an unbounded operator in L^2

- (1) The (weak) maximal closure of $\overline{\partial}$:
  Realize $\overline{\partial}$ as a closed densely defined (maximal) operator

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The L^2-Dolbeault Coholomolgy is defined by

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- (2) The (strong) minimal closure of $\overline{\partial}$: Let $\overline{\partial}_c$ be the (strong) minimal closed L^2 extension of $\overline{\partial}$.

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By this we mean that $f \in \text{Dom}(\overline{\partial}_c)$ if and only if there exists a sequence of compactly supported smooth forms f_ν such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$.

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Closed range property for pseudoconvex domains in $\mathbf{C}^n$

Hörmander 1965

If $\Omega \subset\subset \mathbb{C}^n$ is bounded and pseudoconvex, then

$$H^{p,q}_{L^2}(\Omega) = 0, \qquad q \neq 0.$$

(Kohn) Sobolev estimates for the $\overline{\partial}$-problem

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then

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Ideas of the proof

Use the weight function $t|z|^2$, $t > 0$, to set up the problem in the weighted L^2 space with respect to weights $L^2(\Omega, e^{-t|z|^2})$.

- In Hörmander's case, we first choose $t > 0$ to obtain the L^2 existtence theorem. Set $t = \delta^{-2}$ where δ is the diameter of the domain Ω to obtain the estimates independent of the weights:

$$\|f\|^2 \leq \frac{e\delta^2}{q}(\|\bar{\partial}f\|^2 + \|\bar{\partial}^* f\|^2), \quad f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*).$$

  If $q = n$, this is the Poincaré's inequality.
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Serre Duality Theorem

The classical Serre Duality theorem is a powerful tool in complex analysis. The formulation is similar to the Poincaré's duality.

Theorem (Serre Duality 1955)

Let Ω be a domain in a complex manifold and let E be a holomorphic vector bundle on $\overline{\Omega}$. Let $\overline{\partial}_E$ has closed range in the Fréchet space $C^\infty_{p,q}(\Omega, E)$ and $C^\infty_{p,q+1}(\Omega, E)$. We have $H^{p,q}(\Omega, E)' \cong H_c^{n-p,n-q}(\Omega, E^*)$.

- The classical Serre duality are duality results of Dolbeault coholomology group $H^{p,q}(\Omega, E)$ for E-valued smooth (p, q)-forms with the Fréchet topology and compactly supported smooth E^*-valued forms with the natural inductive limit topology.
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L^2 Serre Duality

Theorem (Chakrabarti-S 2012)

Let Ω be a bounded domain in a complex hermitian manifold of dimension n and let E be a holomorphic vector bundle on $\overline{\Omega}$ with a hermitian metric h. Suppose that $\Box_E$ has closed range on $L^2_{p,q}(\Omega, E)$. Then $\Box_{c,E^*}$ has closed range on $L^2_{n-p,n-q}(\Omega, E^*)$ and $H^{p,q}_{L^2}(\Omega, E) \cong H^{n-p,n-q}_{c,L^2}(\Omega, E^*)$.

- Let $\star_E : C^\infty_{p,q}(\Omega, E) \to C^{n-p,n-q}(\Omega, E^*)$ be the Hodge star operator.

$$\star_E \Box_E = \Box^c_{E^*} \star_E .$$

- This gives the explicit formula:

$$\star_E \mathcal{H}^{p,q}(\Omega, E) = \mathcal{H}^{n-p,n-q}_{c,L^2}(\Omega, E^*).$$

- The theorem follows from the L^2 Hodge theorem.

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The Oka's Lemma

Let $\Omega \Subset \mathbb{C}^n$.

Oka's Theorem

- Suppose Ω is pseudoconvex in $\mathbb{C}^n$. Then there exists a strictly plurisubharmonic exhaustion function.

Bounded plurisubharmonic exhaustion functions

If $\Omega \subset\subset \mathbb{C}^n$ is a bounded pseudoconvex domain with C^2 boundary. Then there exist a defining function r and a positive constant $0 < \eta \leq 1$ such that $\hat{r} = -(-r)^\eta$ is plurisubharmonic on Ω (Diederich-Fornaess 1977).

- There exists a bounded Hölder continuous plurisubharmonic exhaustion function.
- This is also true if the boundary is just Lipschitz (Kerzman-Rosay, Demailly, Harrington).

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Strong Oka's lemma and the Diederich-Fornaess exponent

Let r be a defining function for the pseudoconvex domain Ω such that

$$-i\partial\overline{\partial}\log(-r) \geq 0.$$

Let $\tilde{r} = re^{-t|z|^2}$ for some $t > 0$. Then $\delta = -\tilde{r}$ satisfies

$$i\partial\overline{\partial} - \log\delta = i\partial\overline{\partial} - \log r + it\partial\overline{\partial}|z|^2 \geq t\omega.$$

We say that $\delta = -\tilde{r}$ satisfies the strong Oka's lemma. Let $0 < t_0 \leq 1$.

The following three conditions are equivalent:

- $i\partial\overline{\partial}(\log\delta) \geq it_0\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2}$.
- $i\partial\overline{\partial}(-\delta^{t_0}) \geq 0$.
- For any $0 < t < t_0$, $i\partial\overline{\partial}(-\delta^t) \geq C_t\delta^t(\omega + i\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2})$ for $C_t > 0$.

Suppose that the boundary is C^2. There exists $0 < \eta_0 \leq 1$

$$i\partial\overline{\partial} - \log\delta \geq i\eta_0\frac{\partial\delta\wedge\overline{\partial}\delta}{\delta^2} \quad\Leftrightarrow\quad i\partial\overline{\partial}(-\delta^{\eta_0}) \geq 0.$$

Boundary regularity for the $\overline{\partial}$-Neumann problem

Boas-Straube (Boundary Regularity when $\eta = 1$)

Suppose Ω is a bounded pseudoconvex domain with smooth boundary in $\mathbb{C}^n$ such that there exists a defining function plurisubharmonic on the boundary $b\Omega$. The Bergman projection B and the canonical solution operator $\overline{\partial}^* N$ are exact regular on W^s, $s \geq 0$.

Sobolev estimates for the $\overline{\partial}$-Neumann problem on Lipschitz domains

Suppose Ω is a bounded pseudoconvex domain with Lipschitz boundary in $\mathbb{C}^n$

- The Bergman projection B and the canonical solution operator $\overline{\partial}^* N$ are exact regular on W^ϵ when $\epsilon < \frac{\eta}{2}$. (Berndtsson-Charpentier)
- $N : W^\epsilon_{0,1}(\Omega) \to W^\epsilon_{0,1}(\Omega)$ (Cao-S-Wang).
- B and N are not regular on the Diederich-Fornaess worm domains for some $W^s(\Omega)$ (Barrett).

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Closed range property for non-pseudoconvex domains

- Let $\Omega \Subset X$ is a domain in a complex manifold X and $b\Omega$ satisfies Andreotti-Grauert condition $A(q)$, i.e., the Levi form has at least either $n-q$ positive eigenvalues or $q+1$ negative eigenvalues at each boundary point.
  It follows from Hörmander-Kohn's theory, subelliptic $\frac{1}{2}$ estimates hold and the closed range holds. Furthermore, $H^{p,q}_{L^2}(\Omega)$ is finite dimensional.
- If $\Omega \Subset X$ is an *annulus between two smooth strongly pseudoconvex domains* and $n \geq 3$, i.e.

$$\Omega = \Omega_1 \setminus \Omega_0,$$

  then $\overline{\partial}$ has closed range, and

$$H^{p,q}_{L^2}(\Omega) = 0$$

  if $q \neq 0$ and $q \neq n-1$.
- This result is not true for complex manifold. If $n = 2$ and $q = 1$, there exists an annuli domain with *strongly* pseudoconcave boundary such that $\overline{\partial}$ does not have closed range (Rossi's example).

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Application of L^2 Serre duality

- Let $\Omega \Subset \mathbb{C}^n$ be an annulus between two bounded pseudoconvex domains, i.e.
$$\Omega = \Omega_1 \setminus \overline{\Omega_0},$$
- Suppose that the boundary of Ω_0 is C^2. Then
$$H^{p,q}_{L^2}(\Omega) = 0, \quad 0 < q < n-1.$$
- Suppose that the boundary of Ω_0 is Lipschitz and Ω_1 is smooth. Then
$$H^{p,q}_{W^s}(\Omega) = 0, \qquad s \geq 1$$
if $0 < q < n-1$.
- The boundary of Ω_0 is only Lipschitz smooth. This is proved by the L^2 Serre duality with singular weights δ^t where δ is the distance function to the boundary satisfying the strong Oka's lemma.

Harmonic spaces for $q = n - 1$ on the annulus

Hörmander 2004

Let $\Omega = B_1 \setminus \overline{B}_0$, where B_1 and B_0 are two concentric balls in $\mathbb{C}^n$. Then $\Box$ has closed range and the harmonic space $H^{p,n-1}_{L^2}(\Omega)$ is isomorphic to the Bergman space $H_{L^2}(B_0)$. The harmonic space
$\mathcal{H}^{n,n-1}(\Omega) = \{\sum_j h(\frac{z}{|z|^2}) \star d\bar{z}_j \mid h \in H_{L^2}(B_0)\}$.

Duality between harmonic and Bergman spaces (2011)

Let $\Omega = \Omega_1 \setminus \overline{\Omega_0} \Subset \mathbb{C}^n$ where Ω_1 is bounded and pseudoconvex and $\Omega_0 \Subset \Omega_1$ is also pseudoconvex but with C^2 smooth boundary, then again closed range holds for $q = n - 1$ and

$$H^{n,n-1}_{L^2}(\Omega) \cong H_{L^2}(\Omega_0).$$

If $b\Omega_0$ is not C^2, it is not known if $H^{0,n-1}(\Omega)$ is Hausdorff.

More on the annulus

Let $T \Subset \mathbb{C}^2$ be the Hartogs triangle

$$T = \{(z, w) \mid |z| < |w| < 1\}.$$

Then T is not Lipschitz at the origin.

- Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ such that $\overline{T} \subset \Omega$. Then $H^{0,1}(\Omega \setminus \overline{T})$ is not Hausdorff (Trapani, Laurent-S).
- If we replace H by the bidisc $\triangle^2$, then $H^{0,1}(\Omega \setminus \overline{\triangle^2})$ is Hausdorff since $\triangle^2$ has a Stein neighborhood basis (Laurent-Leiterer).

Chinese Coin Problem

Let B be a ball of radius two in $\mathbb{C}^2$ and $\triangle^2$ be the bidisc. Determine if the L^2 cohomology $H^{0,1}_{L^2}(B \setminus \overline{\triangle^2})$ is Hausdorff.

Outline

Solution to the Chinese Coin Problem

Let $V_1, \ldots, V_n$ be bounded planar domains in $\mathbb{C}$ with Lipschtz boundary and let $V = V_1 \times \cdots \times V_n$.

Theorem (Chakrabarti-Laurent-S)

Let $\tilde{\Omega}$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $V \Subset \tilde{\Omega}$. Let $\Omega = \tilde{\Omega} \setminus \overline{V}$ be the annulus between $\tilde{\Omega}$ and V. Then $H^{0,1}_{L^2}(\Omega)$ is Hausdorff and

- $H^{0,1}_{L^2}(\Omega) = \{0\}$, *if* $n \geq 3$.
- $H^{0,1}_{L^2}(\Omega)$ *is infinite dimensional if* $n = 2$.

Corollary

Let V be the product of bounded planar domains with Lipschitz boundary. Then

$$H^{0,n-1}_{W^1}(V) = \{0\}.$$

Duality between the cohomology on the annuli and the hole

Let $\tilde{\Omega}$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $V \Subset \tilde{\Omega}$. Let $\Omega = \tilde{\Omega} \setminus \overline{V}$ be the annulus between $\tilde{\Omega}$ and V.

Lemma

Then the following are equivalent:

1. $H^{0,1}_{L^2}(\Omega)$ *is Hausdorff.*
2. $H^{n,n}_{c,L^2}(\Omega)$ *is Hausdorff.*
3. $H^{0,n-1}_{W^1}(V) = 0$

- (1) and (2) are equivalent following the L^2 Serre duality. Thus to study the L^2 cohomology of Ω is equivalent to the W^1-estimates for $\bar{\partial}$ on the inner domain V.
- If V is a pseudoconvex domain with C^2 boundary, then (3) holds. For Lipschitz domains, even when V is the bidisc, this is not known!

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Outline

Diederich-Fornaess exponent in complex projective spaces

Let Ω be a pseudoconvex domain in $\mathbb{CP}^n$ equipped with the Fubini-Study metric ω.

Takeuchi's Theorem 1964

The signed distance function $\rho = -\delta$ for Ω satisfies

$$i\partial\overline{\partial} - \log \delta \geq C\omega$$

where $C > 0$. We say that δ satisfies the strong Oka's lemma.

Ohsawa-Sibony 1998

Suppose Ω has C^2 boundary. There exists a positive Diederich-Fornaess exponent $0 < \eta \leq 1$ for the distance function δ under the Fubini-Study metric.

- This is also true for pseudoconvex domains in $\mathbb{CP}^n$ with Lipschitz boundary (Harrington (2015)).

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The $\overline{\partial}$-problem on pseudoonvex domains in $\mathbb{CP}^n$

Let $\Omega \Subset \mathbb{CP}^n$ be pseudoconvex.

- (Takeuchi) Ω is Stein and $H^{p,q}(\Omega) = 0, \ q \neq 0$.
- Using the Bochner-Kodaira-Morrey-Kohn formula, $H^{0,1}_{L^2}(\Omega) = 0$.

Boundary Regularity

Suppose Ω has Lipschitz boundary. Let η be the Diederich-Fornaess exponent with $0 < \eta \leq 1$.

- For $0 \leq p \leq n$, $H^{p,1}_{L^2}(\Omega) = 0$.
- $N : W^{\epsilon}_{0,1}(\Omega) \to W^{\epsilon}_{0,1}(\Omega)$, $\epsilon < \frac{\eta}{2}$. (Berndtsson-Charpentier, Cao-S-Wang)
- Open Question: Can one have $H^{0,1}_{W^s}(\Omega) = 0$ for $s \geq \frac{1}{2}$?
- If yes, this will give closed-range property for $\overline{\partial}_b$ on pseudoconvex boundary in $\mathbb{CP}^n$.
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Levi-flat hypersurfaces in $\mathbb{CP}^n$

Let M be a compact hyper surface in $\mathbb{CP}^n$ such that M divides $\mathbb{CP}^n$ into two pseudoconvex domains. Then M is called *Levi-flat*.
If M is C^1 smooth, then it is foliated by complex submanifolds locally

$$M \cap U = \cup_t \Sigma_t$$

where Σ_t is a complex manifold of dimension $n-1$.

Motivation: Complex foliation and complex dynamics.

- Lins-Neto (1999) There exist no real-analytic Levi-flat hypersurfaces in $\mathbf{C}P^n$, $n \geq 3$.
- Siu (2000) There exist no smooth Levi-flat hypersurfaces in $\mathbf{C}P^n$, $n \geq 3$.
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The $\overline{\partial}$-equation on pseudoconcave domains in $\mathbb{C}P^n$, $n \geq 3$

Let Ω be pseudoconvex in $\mathbf{C}P^n$ with $\overline{\Omega} \neq \mathbf{C}P^n$, where $n \geq 3$. Let

$$\Omega^+ = \mathbf{C}P^n \setminus \overline{\Omega}$$

Cao-S-Wang (2004)

Suppose the boundary $b\Omega$ is $\mathbf{C}^2$. We have

$$H^{0,1}_{W^{1+s}}(\Omega^+) = \{0\}, \qquad 0 \leq s < \eta/2.$$

Cao-S (2007)

Suppose Ω has Lipschitz boundary. We have

$$H^{0,1}_{W^{1+s}}(\Omega^+) = \{0\}, \qquad 0 \leq s < \frac{1}{2}.$$

Corollary:

There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$.

The Diederich-Fornaess exponent and Levi-flat hypersurfaces

(Fu-Shaw 2016)

Let Ω be a bounded Stein domain with C^2 boundary in a complex manifold hermitian M of dimension n. If the Diederich-Fornæss index of Ω is greater than k/n, $1 \leq k \leq n-1$, then Ω has a boundary point at which the Levi form has rank $\geq k$.

- If the Diederich-Fornæss index is greater than $1/n$, then its boundary cannot be Levi flat; and if the Diederich-Fornæss index is greater than $1 - 1/n$, then its boundary must have at least one strongly pseudoconvex boundary point.
- There exists a domain with Levi-flat boundary in a two-dimensional complex manifold with $\eta_0 = \frac{1}{2}$ (Diederich-Ohsawa).
- Adachi-Brinkschulte obtained similar results independently using different methods.

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Proof:

Nemirovski 1999

A Stein domain Ω in a complex manifold with compact Levi-flat boundary does not admit a plurisubharmonic defining function.

proof

Assume that $n = 2$. Suppose there exists a plurisubharmonic function ψ for Ω. Let $\Omega_t = \{\psi < t\}$, $-\epsilon < t \leq 0$. Define

$$F(t) = \int_{b\Omega_t} d^c\psi \wedge dd^c\psi.$$

Then $F(t) \geq 0$ by Stokes's theorem and $F(0) = 0$. For $t \geq s$, we have

$$F(t) - F(s) = \int_{\Omega_t \setminus \overline{\Omega}_s} dd^c\psi \wedge d \wedge dd^c\psi \geq 0.$$

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Hartogs' Triangles in $\mathbb{CP}^2$

In $\mathbb{CP}^2$, we denote the homogeneous coordinates by $[z_0, z_1, z_2]$. On the domain where $z_0 \neq 0$, we set $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$.
Let H^+ and H^- be defined by

$$H^+ = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| < |z_2|\}$$
$$H^- = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| > |z_2|\}$$

$$M = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| = |z_2|\}.$$

$$H^+ \cup M \cup H^- = \mathbb{CP}^2.$$

These domains are called Hartogs' triangles in $\mathbb{CP}^2$. It is not Lipschitz at 0 and it is not foliated near 0.

L^2 theory for $\overline{\partial}$ on Hartongs Triangle

- Both H^+ and H^- are pseudoconvex.
- M is a (non-Lipschitz) Levi-flat hypersurface in $\mathbb{CP}^2$.
- $H^{0,1}_{L^2}(H^+) = 0$. But $H^{1,1}_{L^2}(H^+)$ and $H^{2,1}_{L^2}(H^+)$ are not known, not even the Hausdorff property.

Definition

Let

$$\overline{\partial}_s : L^2_{2,0}(H^+) \to H^{2,1}(H^+)$$

denote the *strong* L^2 closure of $\overline{\partial}$.

- We do not know if $\overline{\partial}_s$ has closed range.
- we do not know if $\overline{\partial} = \overline{\partial}_s$ (weak equals strong).
- $H^{2,1}_{\overline{\partial}_s, L^2}(H^+)$ is infinite dimensional (Laurent-Shaw 2018).

Outline

Non-closed range property for some Stein domain

Theorem (Chakrabarti-S, 2015 Math. Ann.)

There exists a pseudoconvex domain Ω in a complex manifold such that

- *Ω is Stein with smooth (real-analytic) Levi-flat boundary.*
- *Any continuous bounded plurisubharmonic function on Ω is a constant.*
- *$\overline{\partial}$ does not have closed range in $L^2_{2,1}(\Omega)$.*
- *$H^{2,1}_{L^2}(\Omega)$ is non-Hausdorff.*

Let

$$X = \mathbb{CP}^1 \times T$$

be a compact complex manifold of dimension 2 endowed with the product metric where T is the torus.
The domain $\Omega \subset X = \mathbb{C}P^1 \times T$ is defined by

$$\Omega = \{(z,[w]) \in \mathbb{C}P^1 \times T : Rezw > 0\}.$$

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Remarks

- Ω is biholomorphic to a punctured plane $\mathbb{C}^*$ and an annulus. Hence Ω is Stein (Ohsawa 1982).
- $$H^{p,q}(\Omega) = 0, \quad q > 0.$$
- We still do not know if $H^{0,1}_{L^2}(\Omega)$ or $H^{1,1}_{L^2}(\Omega)$ is Hausdorff.
- An earlier example (constructed by Grauert) of a pseudoconvex domain in a a two-tori has been shown with non-Hausdorff property by Malgrange (1975). But the domain is not holomorphically convex (not Stein).

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